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A practical, general-purpose, two-state HLL Riemann solver for hyperbolic conservation laws

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SUMMARY

In this paper, we construct a general-purpose Riemann solver for hyperbolic conservation laws that does not involve extensive characteristic analysis of governing equations but can nevertheless sharply resolve discontinuities. To achieve this goal, we revisit the class of two-state HLL schemes and show that inexpensive, accurate solvers within this class can be designed using only geometric interpretations of the Rankine–Hugoniot conditions. We argue that the small cost and nearly uniform algorithmic complexity of resulting methods make them attractive for quick computations of practical, especially very complex, problems for which more accurate solvers are either not available or their development cost is not justified. Copyright © 2002 John Wiley & Sons, Ltd.

KEY WORDS: Riemann solvers; HLL schemes; hyperbolic conservation laws

1. INSTRUCTIONS

Over the past two decades Godunov-type finite-volume methods have firmly established themselves as the dominant computational approach to solving systems of hyperbolic conservation laws. Underlying the popularity of these methods are their sound mathematical foundation and algorithmic framework, as well as the accuracy and robustness with which the methods resolve both smooth and discontinuous flows. It would be fair to claim that Godunov-type methods contributed significantly to the dramatic spreading of high-quality computations in a wide range of scientific disciplines. From traditional fluid mechanics applications, they have been extended to solve problems in plasma physics, space physics, astrophysics and many other scientific and engineering fields.

As the proliferation of Godunov-type methods into successively more complex physical systems continues, it is becoming apparent that the main obstacle to this process will be the availability of accurate Riemann solvers, algorithms for resolving discontinuous Riemann

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problems. Indeed, in any Godunov-type method, most of the essential blocks needed to create a particular application can be implemented using universal, problem-independent algorithms. Time integration, high-resolution variable reconstruction and even adaptive mesh refinement are but a few examples of operations that, in principle, admit reusable, model-free implementations. However, the Riemann solver, being the only place in the method where problem-specific information is used, cannot generally be a universal algorithm. In order to achieve superior accuracy, the solver must be able to sort out wave motions that arise in discontinuous situations. The best solvers achieve this by using characteristic wave decomposition analysis, which certainly depends on the precise form of the governing equations. The problem is that the complexity of such analysis increases dramatically as the equations become more and more complex. For sufficiently complex systems, carrying out this task may become extremely difficult, impractical and even impossible.

A remedy against this problem would be to use an algorithm that is not tied to specific details of the governing equations. However, many such algorithms, for example the Lax–Friedrichs [1] algorithm, typically have poor accuracy, because they discard most of the available characteristic information. Solver accuracy and universality, therefore, seem to be at odds with each other. The question is whether reasonable trade-offs can be found, i.e. whether it is feasible to design solvers that feature acceptable accuracy yet do not involve extensive characteristic analysis.

An approach that one might take is to embed high-order reconstruction directly into problem-independent solvers. This has been done with the Lax–Friedrichs and local Lax–Friedrichs (also known as Rusanov) solvers [2, 3], and the resulting central schemes were shown to have very good accuracy. However, these scheme still lack an important property of more complex schemes—they do not have built-in mechanisms to identify and resolve isolated discontinuities. This should degrade the schemes' performance around slow moving and, especially, stationary discontinuities.

In this paper, we want to explore another approach to designing accurate, general-purpose Riemann solver algorithms. We will revisit the class of two-state HLL solvers proposed by Harten *et al.* [4] and show that all of the elements needed to construct these solvers can be simply evaluated using only geometric interpretations of the Rankine–Hugoniot conditions in a properly rescaled vector space of conserved variables. We will derive an algorithm that is only loosely tied to particular physical models and is independent of specific characteristic properties of the governing equations. Since the algorithm does not involve extensive characteristic decomposition analysis, its algorithmic complexity is the same for all physical systems. Using minimal input information, it can perfectly resolve all physically admissible discontinuities. This makes the suggested solver an attractive, general-propose solution method for practical, especially very complex, problems.

2. ANALYSIS

Let us consider a general system of conservation laws. In one spatial dimension, a simplification we make strictly for the sake of brevity of our analysis, the governing equations are

$$U_t + F(U)_x = 0 \tag{1}$$

where U is a set of conserved quantities, and $F(U)$ is the flux of these quantities. We will assume that the system is hyperbolic, and that it possesses a scalar, convex entropy function $s(U)$, which satisfies

$$s(U)_t + \sigma(U)_x \leqslant 0 \tag{2}$$

where $\sigma(U)$ is some function called entropy flux. We will further assume that we know and can compute both of these functions. This is not a very restrictive assumption, because entropy functions of most physical systems are known, and their corresponding entropy flux functions can be derived using straightforward algorithms.

Another assumption that we will make in this paper is that, we can somehow bound the eigenvalues of the Jacobian matrix $\partial F(U)/\partial U$. That is, for any conserved state, we can find $\lambda_{\min}(U)$ and $\lambda_{\max}(U)$ such that $\lambda_{\min}(U) \leqslant \partial F(U)/\partial U \leqslant \lambda_{\max}(U)$. We need this assumption to be able to estimate the speed of propagation of disturbances in the system. Since this is required, among other things, to ensure stability of any explicit numerical scheme, the above assumption naturally represents the absolutely minimum amount of characteristic information we need to know about the system. For many systems, it can be easily obtained. We make no additional assumptions about the equations we intend to solve.

In order to solve the above system of equations numerically, we will use the Godunov [5] approach. Since all Godunov-type algorithms are conceptually equivalent, we will consider the simplest of them,

$$U_i(t + \Delta t) = U_i(t) - \frac{\Delta t}{\Delta x}\left[F(U_i(t), U_{i+1}(t)) - F(U_{i-1}(t), U_i(t))\right] \tag{3}$$

in which U_i is the cell-averaged state of conserved variables, $F(\cdot,\cdot)$ is the interface flux between cells i and $i+1$, and Δt and Δx are, respectively, the time increment and the cell size. The interface flux is a function of two conserved states separated by the grid interface. In our highly simplified scheme, these states are simply the average states in cells that share the same interface. In practical schemes, these states are computed using high-resolution reconstruction methods. Wherever these two states come from, the role of the interface flux is to compute a reasonable approximation to the physical flux that should exist at the grid interface between times t and $t+\Delta t$. If the flow is smooth, approximating this flux can be done very accurately, because the two states and the physical fluxes associated with them should be very close to each other in the vector space of conserved variables. If, however, the flow is discontinuous in the neighbourhood of the interface, the states and their fluxes are generally far apart from each other, and one cannot use straightforward interpolation to obtain the interface flux.

In order to resolve discontinuous situations, Godunov-type schemes use Riemann solvers, which sort out the flow of information and obtain physically meaningful fluxes of conserved quantities. Most solvers perform this function using highly detailed characteristic wave decompositions. While this approach certainly produces excellent results, its power is unnecessary most of the time. Let us recall that solutions of systems of hyperbolic conservation laws consist of smooth regions separated by isolated discontinuities. Since smooth regions can be very successfully handled by high-order approximation algorithms, Riemann solvers are needed only around isolated discontinuities. Traditionally, stress in this statement is placed on the word discontinuities. Instead, we want to stress the word isolated, because this shifts focus in Riemann solver design from wave decomposition analysis to capturing of isolated discontinuities. The latter task is significantly simpler than the former one. Of course, our logic fails at

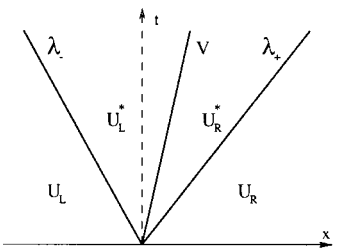

Figure 1. Approximate Riemann problem solution with two intermediate states.

the points of intersection of discontinuities, where neither are solutions smooth nor are discontinuities isolated. The only way to resolve these singular situations is by solving appropriate Riemann problems exactly. However, we can make the following observations. First, even if we can resolve complex wave interactions exactly, the finite-volume method destroys the details of the interactions due to its volume-averaging formalism. The method cannot capture discontinuity interactions on the grid scale. Second, in practical computations, discontinuity interactions are typically infrequent and spatially compact. If needed, their resolution can be improved at little cost using grid adaptation techniques.

A simple framework for resolving isolated discontinuities exists. Let us recall the class of HLL Riemann solvers, in particular, the two-state HLL solver [4]. In this solver, the solution of a Riemann problem is approximated by two intermediate states that link the initial states (see Figure 1). If the intermediate states and their extent are somehow determined, one can use conservation laws to derive the approximate interface flux. In particular, using this framework, one can construct solvers in which the middle wave separating the intermediate states is used to perfectly capture all isolated discontinuities whenever they are present in the flow. The classic HLL paper [4] suggests a two-state solver that has this property. However, its algorithm is non-intuitive, cumbersome and somewhat ambiguous, because the solver contains arbitrary constants. Below, we will show how to close a two-state HLL solver in a substantially simpler manner.

Let us denote the left and right states in the Riemann problem by U_{L} and U_{R}, and the intermediate states in its approximate solution by U_{L}^* and U_{R}^*. Let us also denote by λ_- and λ_+, the lower and upper bounds, respectively, for the smallest and largest signal velocities, and by V, the velocity of the middle wave. If $\lambda_- \geqslant 0$ or $\lambda_+ \leqslant 0$, the interface flux can be trivially upwinded using, respectively, the left or the right state. Let us, therefore, consider the case when $\lambda_- < 0 < \lambda_+$. Then

$$F(U_{\mathrm{L}}, U_{\mathrm{R}}) = \begin{cases} F(U_{\mathrm{L}}) + \lambda_-(U_{\mathrm{L}}^* - U_{\mathrm{L}}) & \text{if } V > 0, \\ F(U_{\mathrm{R}}) + \lambda_+(U_{\mathrm{R}}^* - U_{\mathrm{R}}) & \text{otherwise} \end{cases} \tag{4}$$

and conservation requires that two intermediate states satisfy

$$(V - \lambda_-)U_{\mathrm{L}}^* + (\lambda_+ - V)U_{\mathrm{R}}^* = \lambda_+ U_{\mathrm{R}} - \lambda_- U_{\mathrm{L}} - \Delta F \tag{5}$$

where $\Delta F = F(U_{\mathrm{R}}) - F(U_{\mathrm{L}})$. To solve the above equation for either of the intermediate states and hence to compute the interface flux, we need an additional equation relating U_{L}^* and U_{R}^*. In general, this requires characteristic analysis of the equations. Since we want to avoid

this step, we will, instead, invoke the following heuristic argument. If the Riemann problem contains an isolated discontinuity, and we demand that this wave be captured by the middle wave, the jump between the intermediate states equals $\Delta U = U_R - U_L$. If there is no isolated discontinuity, one cannot determine ΔU^* without fully solving the problem. However, as we argued above, in this case, the flow is likely to be locally smooth, and it would suffice for the solver only to mimic the correct physical behaviour that the conserved variables gradually change their values across the interaction region. This is easily achieved by setting ΔU^* to 0. We can express the entire argument by one *postulated* equation,

$$U_R^* - U_L^* = \alpha(U_L, U_R)(U_R - U_L), \quad 0 < \alpha < 1 \tag{6}$$

where $\alpha \to 1$ if an isolated discontinuity is present, and $\alpha \to 0$ if no discontinuity can be detected. We can think of α as an estimate of the relative strength of the middle wave. With this postulate, we rewrite Equations (4) and (5) and compute the expression for the interface flux:

$$F(U_L, U_R) = \begin{cases} F(U_L) + \dfrac{\lambda_-}{\lambda_+ - \lambda_-}[((1-\alpha)\lambda_+ + \alpha V)\Delta U - \Delta F] & \text{if } \lambda_- < 0,\ V > 0, \\ F(U_R) + \dfrac{\lambda_+}{\lambda_+ - \lambda_-}[((1-\alpha)\lambda_- + \alpha V)\Delta U - \Delta F] & \text{if } \lambda_+ > 0,\ V < 0 \end{cases} \tag{7}$$

This expression is closed if we can estimate the values of λ_-, λ_+, V and α.

Let us start with V, the velocity of the middle wave. In order to estimate it, we recall that the present solver is designed to capture isolated discontinuities. If such a discontinuity is present, and by design, the discontinuity must coincide with the middle wave, it must satisfy the Rankine–Hugoniot conditions,

$$\Delta F - V \Delta U = 0 \tag{8}$$

Geometrically, this means that in the vector space of conserved variables, ΔF is parallel to ΔU. If ΔF is not parallel to ΔU, no value of V solves the above system of equations exactly. However, one can still solve this system in the least-squares sense and interpret resulting V as the velocity of the wave pattern dominating the interaction of two initial states. The more aligned ΔF and ΔU are, the more prominent such a wave pattern should be.

The least-squares solution of Equation (8) in the P-norm is

$$V = \frac{(\Delta U, \Delta F)_p}{\|\Delta U\|_p^2} \tag{9}$$

where $(\Delta U, \Delta F)_p = \Delta U^T P \Delta F$, $\|\Delta U\|_p^2 = \Delta U^T P \Delta U$ and P is a symmetric, positive definite matrix. This matrix is used here to rescale components of ΔU and define dimensionally consistent inner product and vector norm, because conserved variables generally have different dimensions and scales. This matrix can be chosen quite arbitrary; however, the theory of symmetrizability of conservation laws with entropy suggests a very convenient form for P. Since $s(U)$ is a convex entropy function, $\partial^2 s / \partial U^2$ is a positive definite matrix function of U. One can show [6] that there exist

$$P = \int_0^1 \frac{\partial^2 s}{\partial U^2}((1-\theta)U_L + \theta U_R)\,\mathrm{d}\theta \tag{10}$$

such that

$$\Delta W = P\Delta U \tag{11}$$

where $W(U) = \partial s/\partial U$ is a vector of symmetrizing variables. Since these variables are simply the gradient of the entropy function, they are straightforward to compute provided that the entropy function is known. Substituting the above expression into Equation (9), we obtain

$$V = \frac{\Delta W^{\mathrm{T}} \Delta F}{\Delta W^{\mathrm{T}} \Delta U} \tag{12}$$

This expression, which is identical to the one used in the original HLL scheme, has the following properties [7, 4]: its denominator is non-zero for $\Delta U \neq 0$, V is uniformly bounded by the eigenvalues of the Roe matrix (which exists), and V solves Equation (8) exactly if an isolated discontinuity exists. Clearly, Equation (12) meaningfully defines the velocity of some dominant wave. Remarkably, direct computation of P is not needed to compute V.

Let us now note that with such a definition of V,

$$\frac{\|\Delta F - V\Delta U\|_p^2}{\|\Delta F\|_p^2} + \frac{V^2\|\Delta U\|_p^2}{\|\Delta F\|_p^2} = 1 \tag{13}$$

Suppose there is a wave moving with velocity V. Then one can show that $\Delta F - V\Delta U$ contains flux jump contributions from all waves except for the one moving with this velocity. With this observation, we can interpret the first term in the above equation as the relative strength of all such waves and, consequently, the second term as the relative strength of the wave moving with velocity V, the middle wave in our solver.[‡] If we recall that α introduced in Equation (6) also estimates the strength of the middle wave, we can write

$$\alpha(U_{\mathrm{L}}, U_{\mathrm{R}}) = \frac{V^2\|\Delta U\|_p^2}{\|\Delta F\|_p^2} = \frac{(\Delta U, \Delta F)_p^2}{\|\Delta U\|_p^2\|\Delta F\|_p^2} = \cos^2(P^{1/2}\Delta U, P^{1/2}\Delta F) \tag{14}$$

This definition clearly satisfies the requirements of the postulate that we made in Equation (6). It also has an interesting geometric interpretation—it relates the strength of the dominant wave in the Riemann problem to the angular distance between ΔU and ΔF in an appropriately rescaled vector space.

In Equation (14), $\|\Delta F\|_p^2$ cannot be expressed only in terms of ΔF, ΔU and ΔW. Therefore, in order to compute α, one needs to compute matrix P, for example, using Equation (10). However, this operation may be complex and, strictly speaking, unnecessary. Since α provides only an estimate of the relative strength of the dominant wave, there is no reason to compute P exactly. For estimation purposes, it would suffice to take only its dominant, positive (since P is symmetric, positive definite), diagonal part and use it in place of the full matrix. There is also no need to carry out integration along parametric curves; midpoint estimate of the integral or arithmetic average should be quite adequate. Following this reasoning, we used in

[‡]Here we implicitly assumed that the eigenvector corresponding to wave moving with velocity V is not degenerate. The study of degenerate systems is interesting, but we do not consider them in this paper.

our calculations the following expression for P:

$$P = \mathrm{diag}\left[\frac{\partial^2 s}{\partial U^2}\left(\frac{1}{2}(U_\mathrm{L} + U_\mathrm{R})\right)\right] \tag{15}$$

For most systems of equations this matrix is simple and inexpensive to compute.

Example
In MHD equations, expressed in dimensionless variables,

$$U = (\rho, \rho\mathbf{V}, E, \mathbf{B})^\mathrm{T}$$

$$s(U) = -\rho \ln \frac{p}{\rho^\gamma}$$

$$W(U) = \frac{(\gamma-1)\rho}{p}\left(\frac{p}{(\gamma-1)\rho}\left(\gamma - \ln\frac{p}{\rho^\gamma}\right) - \frac{1}{2}V^2, \mathbf{V}, -1, \mathbf{B}\right)^\mathrm{T}$$

$$P(U) \sim \mathrm{diag}\left(\frac{1}{4}V^4 + \frac{\gamma p^2}{(\gamma-1)^2\rho^2}, V_x^2 + \frac{p}{(\gamma-1)\rho}, V_y^2 + \frac{p}{(\gamma-1)\rho}, V_z^2 + \frac{p}{(\gamma-1)\rho}, 1, B_x^2 + \frac{p}{\gamma-1}, B_y^2 + \frac{p}{\gamma-1}, B_z^2 + \frac{p}{\gamma-1}\right)$$

where $p = (\gamma-1)(E - \frac{1}{2}\rho V^2 - \frac{1}{2}B^2)$ is the thermal pressure (not to be confused with matrix P) and γ is the adiabatic constant. In order to obtain similar expressions for the Euler equations, one needs to set B to zero in the above equations and eliminate appropriate entries from U, W and P.

The only problem with α defined in Equation (14) is that it provides no mechanism for discriminating against entropy violating discontinuities. This happens, because this definition is symmetric with respect to left and right states. However, there is a physically meaningful way to break this symmetry. Recall that the governing equations must satisfy the entropy inequality. In the neighbourhood of a discontinuity moving with velocity V, this inequality must satisfy

$$V\Delta s - \Delta\sigma \geqslant 0 \tag{16}$$

where deltas denote jumps across the discontinuity. One can use the inequality as a test of whether an isolated discontinuity violates the entropy condition. If it does, the discontinuity must be broken, for example, by setting α to 0. If it does not, the discontinuity is physically admissible and should be preserved. Therefore, we can redefine parameter α in the following manner:

$$\alpha(U_\mathrm{L}, U_\mathrm{R}) = H(V\Delta s - \Delta\sigma)\frac{(\Delta U, \Delta F)_p^2}{\|\Delta U\|_p^2 \|\Delta F\|_p^2} \tag{17}$$

where $H(x)$ is the Heaviside function, with $H(0) = 1$.

In order to complete the derivation of the interface flux function, we only need to supply an algorithm for computing the bounding characteristic velocities, λ_- and λ_+. If possible, one can use tight bounds of Einfeldt *et al.* [8], but this entails computing appropriate Roe-average eigenvalues. In general, we want to avoid this. Therefore, we will base our estimates only on characteristic velocities of initial states. Note, that we cannot simply use $\lambda_- = \min(\lambda_{\min}(U_L), \lambda_{\min}(U_R))$ and $\lambda_+ = \max(\lambda_{\max}(U_L), \lambda_{\max}(U_R))$, because these velocities may not bound the wave interaction region [9], and V may lay outside of their range. In order to prevent this from happening and properly bound the wave interaction region, we will include V in the definition of the bounding speeds,

$$\lambda_- = \min(V, \lambda_{\min}(U_L), \lambda_{\min}(U_R)) \quad \text{and} \quad \lambda_+ = \max(V, \lambda_{\max}(U_L), \lambda_{\max}(U_R)) \tag{18}$$

We could also define $\lambda_- = \min(0, V, \lambda_{\min}(U_L), \lambda_{\min}(U_R))$ and $\lambda_+ = \max(0, V, \lambda_{\max}(U_L), \lambda_{\max}(U_R))$, as well as $V_- = \min(0, V)$ and $V_+ = \max(0, V)$ and rewrite the interface flux using a single expression,

$$F(U_L, U_R) = \frac{\lambda_+ F(U_L) - \lambda_- F(U_R)}{\lambda_+ - \lambda_-} + \frac{(1-\alpha)\lambda_-\lambda_+ + \alpha(\lambda_- V_+ + \lambda_+ V_-)}{\lambda_+ - \lambda_-} \Delta U \tag{19}$$

Combined with the expressions for V and α, defined, respectively, in Equations (12) and (17), this expression is the final form of the interface flux that we want to propose in this paper.

To conclude this section we want to point out that, with a special, Lax–Friedrichs-type, choice of $\lambda_+ = -\lambda_- = \lambda \equiv \max(|\lambda(U_L)|, |\lambda(U_R)|)$, the above flux function becomes

$$F(U_L, U_R) = \tfrac{1}{2}(F(U_L) + F(U_R)) - \tfrac{1}{2}((1-\alpha)\lambda + \alpha|V|)\Delta U \tag{20}$$

This equation defines a central, Lax–Friedrichs-type, flux function, whose dissipation coefficient is designed to vanish in the presence of an isolated discontinuity. This function is certainly more diffusive than the one defined in Equation (19), but it can nevertheless perfectly capture discontinuities.

3. RESULTS

We will demonstrate the flexibility and the accuracy of the method in one dimension using test problems from gas dynamics, ideal magnetohydrodynamics (MHD) and relativistic hydrodynamics (RHD). In Figures 2–4, we show density distributions for three test problems that were computed with first-order implementations of, respectively, the proposed solver (labeled New), the Roe solver (labeled Roe), the HLLE solver (labeled HLLE) and the Lax–Friedrichs solver (labeled LxF). This particular set of solvers is chosen, because the algorithmic complexity of the new solver is close to the complexity of the HLLE and the Lax–Friedrichs solvers, while its accuracy, as graphically demonstrated, is comparable to the accuracy of the very accurate Roe solver. To assess overall accuracy of the solvers we also show exact solutions.

Figure 2 shows density distributions in the Sod [10] problem. The solutions were computed with 200 points in a moving frame of reference in which the contact discontinuity is stationary. We chose this frame of reference, because in it the main strength of the proposed solver is

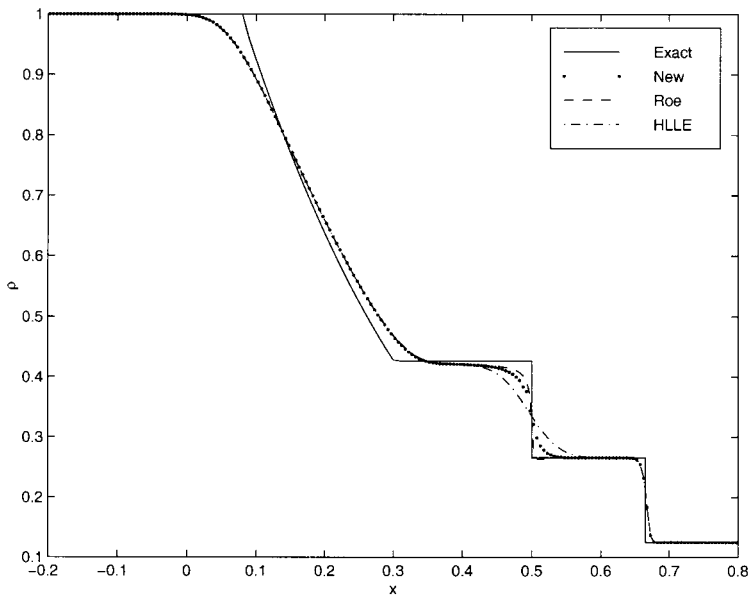

Figure 2. Sod problem.

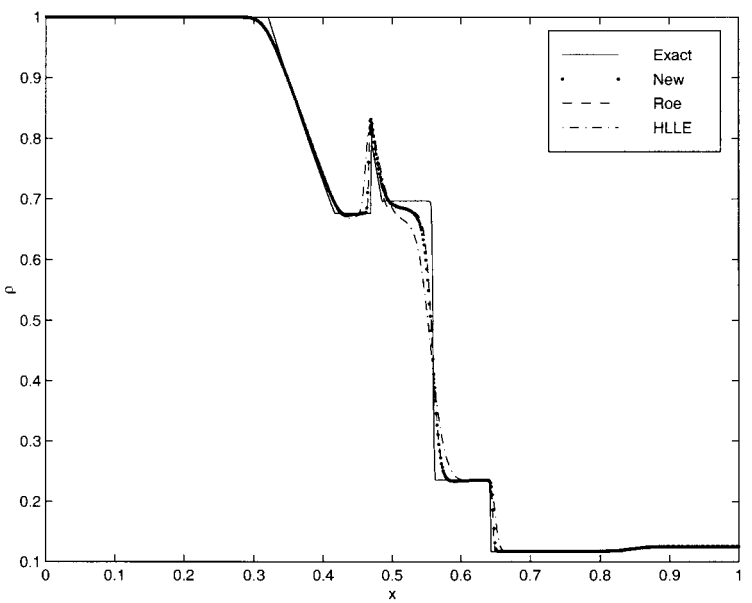

Figure 3. Brio–Wu MHD problem.

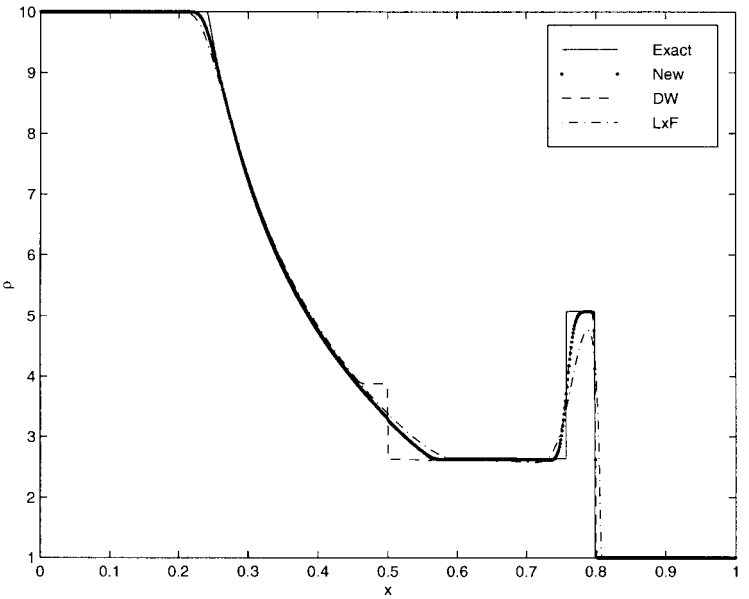

Figure 4. Hawley *et al.* RHD problem.

particularly evident. We can clearly see that the solution generated by our solver is comparable to the one generated by the Roe solver and is noticeably better than the one computed by the HLLE solver. This is quite remarkable given the fact that both the HLLE and Roe solvers use precise characteristic information, and our solver does not. Evidently, the solver's ability to detect isolated discontinuities enhances its accuracy.

Figure 3 shows density distributions in the Brio–Wu [11] MHD shock tube problem computed with 800 points. Again, we can see that the new solver produces results that are close to those produced by the Roe solver and are considerably more accurate than HLLE results. This is because the solver can recognize all physical discontinuities in this problem, while the HLLE solver sees only the fastest left- and right-going waves.

Figure 4 shows density distributions in the Hawley *et al.* [12] relativistic hydrodynamics problem computed with 1000 points. In this problem, the Dai–Woodward (DW) solver is used instead of the Roe solver. The DW solver is essentially the exact solver with the only difference that it uses shock wave expressions to compute both shocks and expansion waves. This can lead to entropy violating solutions, and Figure 4 clearly demonstrates that the first-order version of the DW scheme does violate the entropy condition. As we can observe, except for the region in which the DW solver violates the entropy condition, DW results and results of the new solver are indistinguishable. This is very encouraging, because due to inherent complexity of the RHD equations, the iterative DW solver is substantially more expensive than the new solver. Actually, for RHD equations, which involve non-analytic transformations between different variables, the complexity of the proposed solver is close to that of the local Lax–Friedrichs solver. Figure 4 shows that despite similar complexity, these two solvers feature very different accuracy, with the Lax–Friedrichs solver clearly being inferior.

4. DISCUSSION AND CONCLUSIONS

In this paper, we presented a simple, general-purpose, two-state HLL Riemann solver for Godunov-type simulations of arbitrary systems of hyperbolic conservation laws. The main difference between the proposed solver and the original HLL solver is in the treatment of the middle wave separating two intermediate states. We believe that our algorithm is cleaner and more intuitive. The algorithmic complexity of the solver is independent of the specific details of the governing equations; therefore, it may prove to be useful for simulations of very complex physical systems, for which conventional high-quality solvers may not be available or whose development time may not be justified. The latter situation may occur in rapid, proof-of-concept computations in which overall turnaround time is more crucial than the highest achievable accuracy. In such situations having a simple, yet reasonably accurate solver should be adequate. The solver presented in this paper can serve this purpose.

In one dimension, our solver generates results whose accuracy matches the accuracy of results obtained with the most sophisticated solvers. We find this quite remarkable, given the fact that the solver does not involve any equation-specific characteristic decomposition analysis.

In higher dimensions, our preliminary numerical experiments indicate that the accuracy of the first-order version of the solver slightly degrades compared to the accuracy of the best solvers; however, the loss of accuracy becomes less pronounced in higher order implementations of the solver. The latter finding is very encouraging, because the proposed solver is specifically intended to be used in high-resolution numerical schemes. We believe that the main source of the solver's accuracy loss in higher dimensions is the sensitivity of its discontinuity-detection mechanism to non-alignment between discontinuities and computational grid lines. Because of this, we suggest that in multi-dimensional simulations, the solver will be most efficiently used in codes that involve adaptive, fully unstructured grids that can align themselves with discontinuity fronts.

Clearly, more extensive experimentation is required to fully evaluate the performance of the proposed solver. Specifically, a study of the solver's accuracy in high-resolution schemes is needed. Our own simulations show that the solver performs acceptably in such schemes; however, it may generate small high-frequency oscillations around some discontinuities. This phenomenon is more pronounced with highly accurate characteristic variable reconstruction schemes and is less visible in simpler schemes that, for example, non-aggressively reconstruct primitive solution variables. There is an apparent imbalance between wave resolution properties of reconstruction schemes and of the solver itself. This needs to be explored in more detail. Perhaps, it is unwise to use a highly simplified solver such as the one we have proposed with the best reconstruction schemes. After all, if full characteristic reconstruction is possible, so should be the building of a characteristic information-based Riemann solver. The present solver is proposed precisely for situations when this is not possible or practical.

Our insights are both interesting and promising. HLL framework appears to provide a low-cost, straightforward method to build a reasonably accurate solution algorithm for arbitrary systems of conservation laws. Certainly, the method proposed in this paper will not become the method of choice for simple, extensively studied systems of equations such as the Euler equations. However, it may find use in computations of more complex system of equations. Whether the method will indeed become useful to practical applications is a subject of future research.

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